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SINGULAR SOLUTIONS OF THE PROBLEMS OF THE NONLINEAR THEORY OF ELASTIC DISLOCATIONS

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To use the Volterra theory of dislocations (the elastic models of structural defects) for describing of mechanical phenomena in the dislocation kernel one should take account into nonlinear effects, and consequently apply the tools of the nonlinear theory of elasticity. In the present work, using as an example the problems of wedge disclination and screw dislocation we show that allowance for physical and geometrical nonlinearity in problems on the equilibrium of an elastic body with an isolated defect can produce qualitatively new results as compared to the linear theory. One of the results is the possibility of existence of the so-called "singular" solution that describes formation of a cavity along the defect axis in a continuous body. An integral relation is formulated, which is useful for analyzing the possibility of existence of the singular solution, as well as for determining the dependence of the radius of the resulting cavity on the defect characteristics. The ranges of parameters at which the singular solution exists are determined and the sizes of the resulting cavities are calculated for particular families of nonlinearly elastic potentials. It is established that the singular solution is energetically more profitable. Moreover, its existence is not a consequence of the violation of the Hadamard inequality.

1. Let us consider the deformation of a continuous medium:

$$R = R(r), \quad \Phi = æ\varphi, \quad Z = z. \tag{1.1}$$

Here r, φ, and z are the cylindrical coordinates in the reference configuration of the elastic body; R, Φ, and Z are the cylindrical coordinates in space; $æ$ is a positive constant.

Transformation (1.1) describes [1] the distribution of displacements in a circular cylinder with wedge disclination. If $æ < 1$, a wedge with an aperture angle of $2\pi(1-æ)$ is inserted in the cylinder, which was preliminarily cut by a half-plane $\varphi = 0$. If $æ > 1$, then a sector $2\pi æ^{-1} < \varphi < 2\pi$ is extracted from the cylinder and its edges are joined. Radial displacements of the cylinder points are described by the function $R(r)$.

The geometric characteristics of the strain (1.1) (the strain gradient $\mathbf{C}$, the Cauchy–Green strain measure $\mathbf{G}$, the left tensor of distortions $\mathbf{U}$, and the tensor of rotation $\mathbf{A}$ [2, 3]), corresponding to transformation (1.1), are defined by the expressions

$$\mathbf{C} = R'\,\mathbf{e}_r\mathbf{e}_R + æ\frac{R}{r}\mathbf{e}_\varphi\mathbf{e}_\Phi + \mathbf{e}_z\mathbf{e}_Z, \quad \mathbf{G} = \mathbf{C}\cdot\mathbf{C}^T = R'^2\,\mathbf{e}_r\mathbf{e}_r + (æ\frac{R}{r})^2\mathbf{e}_\varphi\mathbf{e}_\varphi + \mathbf{e}_z\mathbf{e}_z,$$
$$\mathbf{U} = \mathbf{G}^{1/2} = R'\,\mathbf{e}_r\mathbf{e}_r + æ\frac{R}{r}\mathbf{e}_\varphi\mathbf{e}_\varphi + \mathbf{e}_z\mathbf{e}_z, \quad \mathbf{A} = \mathbf{U}^{-1}\cdot\mathbf{C} = \mathbf{e}_r\mathbf{e}_R + \mathbf{e}_\varphi\mathbf{e}_\Phi + \mathbf{e}_z\mathbf{e}_Z, \tag{1.2}$$

where $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z$ and $\mathbf{e}_R, \mathbf{e}_\Phi, \mathbf{e}_Z$ are the unit vectors of cylindrical coordinates in the reference and real configurations respectively: hereafter a prime denotes differentiation with respect to variable r.

Confining our consideration to incompressible materials, we define the function $R(r)$ from the incompressibility condition [2]

$$\det \mathbf{C} = 1$$

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or

$$\frac{\text{æ}\, R\, R'}{r} = 1\,, \tag{1.3}$$

whence we find

$$R(r) = \sqrt{(r^2 + C^2)/\text{æ}} \tag{1.4}$$

(C is an integration constant).

Following the scheme used in [4], we dwell on two cases of generation of wedge disclination: 1) a cylinder remains continuous after deformation, 2) a continuous cylinder becomes hollow. Following the terminology used in [5], the solutions of the first problem and the second problems will be called regular and singular solutions respectively.

It follows from (1.4) that a zero value of the constant C corresponds to the regular solution, whereas in the singular case this constant is proportional to the radius of the resulting cavity

$$C = \sqrt{\text{æ}}\, R(0).$$

Let the elastic potential $W(\mathbf{C})$ of an incompressible material be specified. The derivative $\partial W/\partial \mathbf{C}$ is denoted by $\mathbf{S}$. Then the governing relation for the Piola stress tensor $\mathbf{D}$ will take the form [2]

$$\mathbf{D} = -p\,\mathbf{C}^{-T} + \mathbf{S} \tag{1.5}$$

(p is the pressure function).

Taking account of (1.2) and (1.5), the vector equation of equilibrium $\operatorname{div}\mathbf{D} = 0$ is reduced to one relation

$$p' = R'\left(S'_{rR} + \frac{1}{r}\,(S_{rR} - \text{æ}\, S_{\varphi\Phi})\right). \tag{1.6}$$

Here $S_{rR} = \mathbf{e}_r \cdot \mathbf{S} \cdot \mathbf{e}_R$; $S_{\varphi\Phi} = \mathbf{e}_\varphi \cdot \mathbf{S} \cdot \mathbf{e}_\Phi$.

The first boundary condition for Eq. (1.6) is the absence of stresses at the outer lateral surface of the cylinder:

$$\mathbf{e}_r \cdot \mathbf{D} = 0 \quad \text{at} \quad r = r_1 \tag{1.7}$$

(r_1 is the cylinder radius prior to deformation). Taking account of (1.5), condition (1.7) can be presented in the form

$$p(r_1) = R'(r_1)\, S_{rR}(r_1)\,. \tag{1.8}$$

The second boundary condition for the regular solution is the continuity condition $R(0) = 0$ or, with account of (1.4), $C = 0$. For the singular solution the second boundary condition is the absence of stresses in a deformed state at the surface of the resulting cavity:

$$\mathbf{e}_R \cdot \mathbf{T} = 0$$

($\mathbf{T}$ is the Cauchy stress tensor). With the aid of the relations connecting the Cauchy $\mathbf{T}$ and Piola $\mathbf{D}$ stress tensors [2], this condition in terms of the reference configuration is written as

$$\lim_{r_0 \to 0}\,(p\,(r_0) - R'\,(r_0)\, S_{rR}\,(r_0)) = 0\,. \tag{1.9}$$

For further transformations we use the dimensionless quantities

$$r^* = r/r_1\,, \quad R^* = R/r_1\,, \quad C^* = C/r_1\,, \quad r_1^* = 1\,,$$

where the asterisk will be omitted.

Taking account of the equality

$$S'_{rR} + \frac{1}{r}\, S_{rR} = \frac{1}{r}\,\frac{d}{dr}\,(r\, S_{rR})\,,$$

we write the solution of Eq. (1.6) as

$$p(r) = p(1) + \int\limits_1^r \left(\frac{R'}{r} \frac{d}{dr} (r\, S_{rR}) - \frac{R'}{r} æ\, S_{\varphi\Phi} \right) dr\,.$$

Integrating by parts the first summand of the integrand and taking into account boundary condition (1.8), we find

$$p(r) = R'(r)\, S_{rR}(r) - \int\limits_1^r \left[r S_{rR} \left(\frac{R'}{r} \right)' - \frac{R'}{r} æ\, S_{\varphi\Phi} \right] dr\,.$$

Substituting the expression obtained for $p(r)$ into (1.9), we obtain

$$\int\limits_0^1 \left[\left(\frac{R'}{r} \right)' r\, S_{rR} + æ \frac{R'}{r} S_{\varphi\Phi} \right] dr = 0\,.$$

The last expression can be additionally transformed with the use of Eqs. (1.4) and (1.5). We obtain finally

$$\int\limits_0^1 \frac{1}{\sqrt{r^2 + C^2}} \left(-R'^2\, S_{rR} + S_{\varphi\Phi} \right) dr = 0\,. \tag{1.10}$$

Relation (1.10) can be used for determining the dependence of the constant C on the disclination parameter $æ$ after concrete definition of the law of state of a nonlinearly elastic body (for selecting the potential W).

Relations (1.1) describe plane deformation of a continuous medium. Therefore, without loss of generality one can consider the elastic potential W as a function of the only value I_1, i.e., the first invariant of the Cauchy strain measure $\mathbf{G}$ [2]. Then the intergand in (1.10) can be written as

$$(1 - R'^4)\, \frac{1}{r} \frac{\partial W}{\partial I_1}\,. \tag{1.11}$$

Assuming [2, 3] that $\partial W/\partial I_1 > 0$, one can easily check that the singular solution can exist only for the positive disclination ($æ < 1$). Actually, it follows from (1.3) and (1.4) that if $æ > 1$, R' is always smaller than unity, and therefore, the integrand (1.11) does not change sign at the segment [0,1] and integral (1.10) cannot become zero.

Integrability of the integrand is the necessary condition of the equality (1.10). Put $\partial W/\partial I_1 = \omega\,(I_1)$. Lest the integrand have a nonintegrable singularity, $\omega\,(I_1)$ should tend to zero no slower than r^α, where α is a positive constant. Since $I_1 \to \infty$ when $r \to 0$, as $1/r^2$, $\omega\,(I_1)$ should tend to zero as $I_1 \to \infty$ no slower than $I_1^{-\beta}$ ($\beta = \alpha/2 > 0$), and the potential W as $I_1 \to \infty$ should not grow faster than $I_1^{1-\beta}$, i.e.,

$$W < \text{const}\, I_1^{1-\beta}\,, \quad I_1 \to \infty\,, \quad \beta > 0\,. \tag{1.12}$$

The elastic potential is often written as a function of the first invariant of the left tensor of distortions $\mathbf{U}$:

$$W = W(j_1)\,, \quad j_1 = \mathrm{tr}\,(\mathbf{U})\,. \tag{1.13}$$

Taking account of the equalities

$$\mathbf{S} = \partial W/\partial \mathbf{C} = (\partial W/\partial \mathbf{U}) \cdot \mathbf{A} = (\partial W/\partial j_1)\, \mathbf{A}\,,$$

we write the integrand in (1.10) as

$$(1 - R'^2)\, \frac{1}{\sqrt{r^2 + C^2}}\, \partial W/\partial j_1\,. \tag{1.14}$$

The nonintegrable singularity in (1.14) can arise due to the multiplier $\partial W/\partial j_1$ which will be denoted by $\omega(j_1)$. Since $j_1 \to \infty$ when $r \to 0$, as $1/r$, the function $\omega(j_1)$ when $j_1 \to \infty$ should not grow faster than $j_1^{1-\alpha}$, $\alpha > 0$. The function of the specific potential energy W when $j_1 \to \infty$ should not, in its turn, increase faster than $j_1^{2-\alpha}$. Reasoning from this we conclude that the necessary condition of existence of singular solutions for the potentials of the type (1.13) has the form

$$W < \mathrm{const}\, j_1^{2-\alpha}, \quad j_1 \to \infty, \quad \alpha > 0.$$

2. As an example we consider a family of nonlinearly elastic potentials studied in [6]:

$$W = \mu\,(I_1 - 3)^{\nu} \tag{2.1}$$

(ν and μ are the material constants). If $\nu = 1$ potential (2.1) coincides with the potential of neo-Hooke's material (Treloar's material) [2].

From (1.12) we obtain immediately a restriction on the constant ν: singular solutions can exist only if $\nu < 1$.

The expression for the tensor $\mathbf{S}$ in this case becomes

$$\mathbf{S} = 2\,\mu\,\nu\,(I_1 - 3)^{\nu-1}\,\mathbf{C},$$

and Eq. (1.10) is written as follows

$$\int_0^1 \left|\frac{R'^2 - 1}{R'}\right|^{2(\nu-1)} \frac{1 - R'^4}{r}\, dr = 0. \tag{2.2}$$

It follows from (2.2) that singular solutions can exist only if $\nu > 0$, otherwise the integrand will have the nonintegrable singularity at the point

$$r_C = C\left(\frac{æ}{1 - æ}\right)^{1/2}.$$

As r_C we denote the point at which the quantity $(I_1 - 3)$ becomes zero. Combining the restrictions obtained for the material constant ν, we arrive at the conclusion that singular solutions can exist only for values of ν from the range (0,1).

If $\nu = 0.5$, Eq. (2.2) reduces to the relation

$$\int_0^1 \frac{1 + R'^2}{\sqrt{r^2 + C^2}}\,\mathrm{sign}\,(1 - R'^2)\, dr = 0, \tag{2.3}$$

where

$$R'^2 = \frac{r^2}{æ\,(r^2 + C^2)}.$$

Integral (2.3) can be calculated analytically. The equation for determining the dependence of C on æ becomes

$$(æ + 1)\,\ln\left(\frac{1 + æ + 2\sqrt{æ}}{1 - æ}\,\frac{C}{1 + \sqrt{1 + C^2}}\right) - 2\sqrt{æ} + \frac{1}{\sqrt{1 + C^2}} = 0.$$

The calculations performed have shown that over a sufficiently wide range of the disclination parameter the dependence of C on æ can be considered linear:

$$C \approx 0.85\,(1 - æ).$$

The dependence of the cavity radius C on the material constant ν at fixed value of the disclination intensity $æ = 0.9$ is plotted in Fig. 1.

Materials of the type (2.1) satisfy Hadamard's inequality for all values of the parameter $\nu \geqslant 0.5$ [6]. This means that appearance of singular solutions does follow from the violation of Hadamard's inequality.

The formula

$$\Pi = 2\pi \int_0^1 r\, W(r)\, dr \tag{2.4}$$

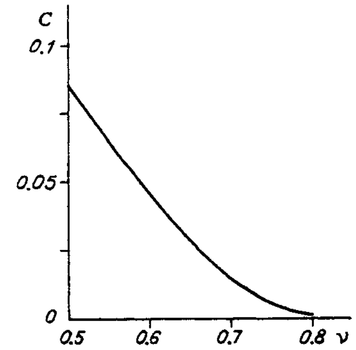

Fig. 1

determines the quantity of specific (per unit length) elastic energy of a deformed cylinder. Calculations with using (1.2), (1.4), and (2.1) for $\nu = 1$ show that the energy is expressed by the relation ($æ < 1$) in the regular

$$\Pi = \mu \pi r_1^2 \frac{1 - æ}{\sqrt{æ}},$$

and in the singular cases

$$\Pi = \mu \pi r_1^2 \frac{1 - æ - æ C^2}{\sqrt{æ}\sqrt{1 + C^2}}.$$

One can easily see that the elastic energy of a hollow cylinder is always less than that of a solid one. This means that the singular solution is more profitable energetically. Similar results are obtained for the other values of ν from the range (0,1).

Now let us consider a nonlinearly elastic potential of the form [7]

$$W = 2\mu \, \mathrm{tr} \, (\mathbf{U}^m - \mathbf{E})/m^2 \tag{2.5}$$

(μ and m are the material constants). If $m = 1$, potential (2.5) coincides with the Bartenev–Khazanovich potential; if $m = 2$, it is reduced to the neo-Hooke's potential [2].

Using the relation $\mathbf{S} = \partial W/\partial C = (\partial W/\partial U) \cdot \mathbf{A}$ and calculating the derivative $\partial W/\partial \mathbf{U} = m\mathbf{U}$, we find the tensor $\mathbf{S}$:

$$\frac{1}{2\mu m} \mathbf{S} = R'^{(m-1)} \mathbf{e}_r \mathbf{e}_R + \left(æ \frac{R}{r} \right)^{(m-1)} \mathbf{e}_\varphi \mathbf{e}_\Phi + \mathbf{e}_z \mathbf{e}_Z .$$

With allowance for the obtained representation of the tensor $\mathbf{S}$ Eq. (1.10) is transformed to the form

$$\int_0^1 \frac{1}{\sqrt{r^2 + C^2}} \left(- R'^{(m+1)} + \left(æ \frac{R}{r} \right)^{(m-1)} \right) dr = 0$$

or

$$\int_0^1 \left[\left(\frac{r^2}{(r^2 + C^2)\, æ} \right)^{(m+1)/2} - \left(\frac{(r^2 + C^2)\, æ}{r^2} \right)^{(m-1)/2} \right] dr = 0 . \tag{2.6}$$

Integral (2.6) is convergent when $-2 < m < 2$.

When $m = 1$ (the Bartenev–Khazanovich material), integral (2.6) is calculated analytically. The equation for determining the radius of the resulting cavity is as follows

$$(1 - æ) \ln \left(\frac{1 + \sqrt{1 + C^2}}{C} \right) = \frac{1}{\sqrt{1 + C^2}} .$$

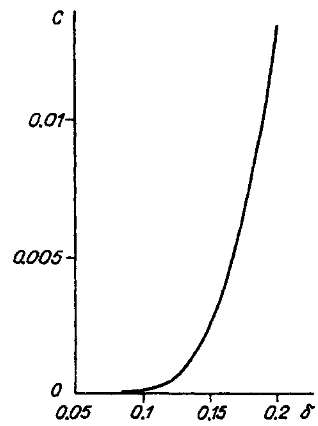

Fig. 2

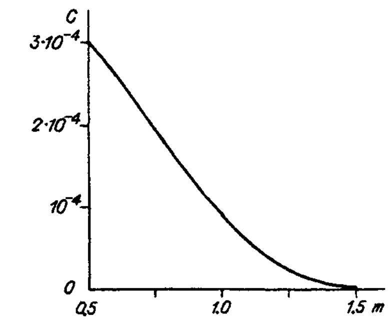

Fig. 3

The dependences of the radii of resulting cavities on the intensity of disclination (at $m = 1$) and on the material constant m (for $æ = 0.9$) are plotted in Figs. 2 and 3 respectively.

3. Deformation of a continuous medium of the form

$$R = R(r), \quad \Phi = \varphi, \quad Z = \frac{b}{2\pi}\varphi + z \tag{3.1}$$

describes the onset of screw dislocation in a cylinder, i.e., cutting of the cylinder by a half-plane $\varphi = 0$ and shear of the cut banks in parallel to the cylinder axis by the value b called the Burgers dislocation tensor. The constant $a = b/2\pi$ will be called the dislocation parameter.

The geometric characteristics corresponding to the transformation (3.1) are defined by the expressions

$$\begin{gathered}
\mathbf{C} = R'\,\mathbf{e}_r\mathbf{e}_R + \frac{R}{r}\,\mathbf{e}_\varphi\mathbf{e}_\Phi + \frac{a}{r}\,\mathbf{e}_\varphi\mathbf{e}_Z + \mathbf{e}_z\mathbf{e}_Z\,, \\
\mathbf{G} = R'^2\,\mathbf{e}_r\mathbf{e}_r + \frac{R^2 + a^2}{r^2}\,\mathbf{e}_\varphi\mathbf{e}_\varphi + \frac{a}{r}\,(\mathbf{e}_\varphi\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\varphi) + \mathbf{e}_z\mathbf{e}_z\,, \\
\mathbf{U} = R'\,\mathbf{e}_r\mathbf{e}_r + \frac{r}{\sqrt{(R+r)^2+a^2}}\Big(\frac{R^2+a^2+Rr}{r^2}\,\mathbf{e}_\varphi\mathbf{e}_\varphi + \frac{a}{r}\,(\mathbf{e}_\varphi\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\varphi) + \frac{R+r}{r}\,\mathbf{e}_z\mathbf{e}_z\Big), \\
\mathbf{A} = \mathbf{e}_r\mathbf{e}_R + \frac{R+r}{\sqrt{(R+r)^2+a^2}}\,(\mathbf{e}_\varphi\mathbf{e}_\Phi + \mathbf{e}_z\mathbf{e}_Z) + \frac{a}{\sqrt{(R+r)^2+a^2}}\,(\mathbf{e}_\varphi\mathbf{e}_Z - \mathbf{e}_z\mathbf{e}_\Phi)\,.
\end{gathered} \tag{3.2}$$

Applying the approach used above for the problem of disclination, we obtain the following results. The function of radial displacement of a point $R(r)$ (for incompressible materials) is of the form

$$R(r) = \sqrt{r^2 + C^2}\,.$$

The equilibrium equations are reduced to the relation

$$p' = R'\left(S'_{rR} + \frac{1}{r}\,(S_{rR} - S_{\varphi\Phi})\right).$$

The equation for determining the dependence of the cavity radius on the dislocation parameter a formally has the same form as (1.10), as in the problem of disclination.

For materials of the Zubov–Rudev type (2.1) the components of the tensor $\mathbf{S}$ with account of (3.2) are written as

$$S_{rR} = \mu\,\nu\,(I_1 - 3)^{\nu-1}\,R'\,, \qquad S_{\varphi\Phi} = \mu\,\nu\,(I_1 - 3)^{\nu-1}\,(R')^{-1}\,.$$

where

$$I_1 - 3 = \frac{C^4 + a^2 (r^2 + C^2)}{(r^2 + C^2) r^2}.$$

Equation (1.10) with the use of the above introduced dimensionless variables becomes

$$\int_0^1 \left[\left(\frac{C^4 + a^2 (r^2 + C^2)}{(r^2 + C^2) r^2} \right)^{\nu - 1} \frac{2 r^2 + C^2}{r (r^2 + C^2)^2} \right] dr = 0. \qquad (3.3)$$

One can easily see that the integrand in (3.3) is strictly greater than zero at the interval (0,1). Thus, integral (3.3) cannot become zero, and therefore, the problem of equilibrium of a cylinder with screw dislocation does not possess a singular solution for Zubov–Rudev materials.

The use of potentials of the form (2.5) results in a significant complication of the problem of screw dislocation. This is due to the fact that to raise to an arbitrary nonintegral power the tensor **U** of the type (3.2) it should first be brought to the principal axes.

In a particular case ($m = 1$) this problem is eliminated off; integral (1.10) is calculated analytically and the equation for determining the constant C takes the form

$$\ln \frac{(1 + \sqrt{1 + C^2})(1 + \sqrt{1 + a^2/C^2})}{\sqrt{1 + C^2} + \sqrt{a^2 + (1 + \sqrt{a^2 + C^2})^2}} - \frac{1}{\sqrt{1 + C^2}} = 0.$$

It is evident from the calculations that in a sufficiently wide range of the dislocation parameter the dependence of C on a can be considered linear:

$$C \approx 0.231\, a, \quad a \in (0.0001, 0.1).$$

The calculations of energy by the formula (2.4) have shown that in the whole range of the problem parameters the energy of a singular solution is lower than the appropriate energy of a regular solution. Therefore, a singular solution can also be considered as energetically more preferable in this case.

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